Structural Optimization and Design Based on a Reliability Design Criterion

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The classical approach to design using a safety factor to represent conservatism often results in unnecessary weight and cost. A rational basis for optimization studies can be obtained by relating safety factor to reliability. The required statistics of the safety factor can be generated by Monte-Carlo and linear perturbation methods. Application of the perturbation method is discussed herein for assignment of design allowables, selection of design materials, and sizing structures for minimum weight.

Nomenclature

A	= area
c	= constant
C	= confidence
d	= constant
E	= modulus of elasticity
$E\{X\}$	= expected value of X
F_c	= compressive allowable stress
F_t	= tensile allowable stress
M	= implied sample size
N	= number of calculations
P	= applied load
p	= variable
R	= reliability
SF	= safety factor
t	= thickness
W	= structural weight
σ	= standard deviation
$\sigma_{p_j p_k}$	= covariance
μ	= mean value
η	= number of standard deviations
α	= thermal coefficient of expansion, in./in./°F
ΔT	= temperature rise
ϵ	= allowable tensile strain
ν	= Poisson's ratio

Superscripts and Subscripts

*	= reference value
SF	= safety factor
mag	= magnesium
asc	= ascent heat shield
stiff	= stiffener
H/S	= re-entry heat shield
long	= longitudinal
lat	= lateral

Introduction

THE structural design of a space vehicle requires the integration of a complex array of variables. A wide variety of structural materials is used to sustain the severe aerodynamic and thermodynamic environments of space flight, which require characteristics of materials never before used for obtaining structural integrity. As space flight experience has been acquired, ability to describe ascent, space, and re-entry conditions mathematically has improved rapidly, so that flight environments and resulting performance can be defined by a complexity of equations relating the multitude of variables involved in the determination of a structural design.

The classical approach to complex structural design problems involves the use of a safety factor that represents the designer's conservatism. The need for conservatism results from uncertainties in defining flight conditions, randomness of the variables defining the environments or the resistance of the vehicle thereto, and the inability to calculate or measure performance under the expected environments due to lack of theoretical or experimental development. This approach, using an arbitrary estimate of conservatism, often results in unnecessary weight and cost, particularly for combined environments. The multiplicity of ill-defined critical design conditions prompts the designer to treat each condition conservatively, resulting in addition and compounding of conservatisms. The accepted means of reducing conservatism is by excessive and costly physical testing and by development of sophisticated mathematical analyses with elaborate test substantiation. Otherwise, the selection of a safety factor represents an estimate of the over-all effect of the uncertainties and randomness involved in the design parameters. Such an estimate, called "design practice," is indeed questionable in the design of space vehicles.

An alternative approach consists of defining a safety factor as a statistical variable and relating it explicitly to a definition of the uncertainties and randomness of the design variables. The safety factor can then be related to the desired reliability with an associated confidence level. Such a method should take into account all natural variations of the parameters, so that one can define which design parameters are significant with respect to safety factor or the variable of interest, as well as the variation of safety factor due to each design variable. In this manner the significant design parameters are defined, and the necessary analytical and experimental development programs can be executed at minimum cost. Similarly, during the detail design of the structure a true structural optimization can then be effected.

Relation between Safety Factor and Reliability

When safety factor is related to statistical variation in performance under given environment, the specification of a safety factor must be made on the basis of statistical considerations. By tradition, the structural design evaluates a design on the basis of a safety factor that expresses the relation between an allowable stress and working stress. Therefore, in a statistical approach it is of interest to consider the safety factor to be itself a random variable and to determine probability density of the quotient (safety factor) in terms of the statistics of the individual stresses.

Mathematical difficulties arise when defining the distribution of the quotient. It is shown in the Appendix, for in-

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stance, that the distribution function of the quotient of two normally distributed random variables is a Cauchy distribution function, which has no standard deviation. A mathematical means of circumventing this problem is by truncating the working stress at $\mu \pm \eta\sigma$ so that a mean and variance can be defined and the safety factor can then be considered to be an approximately normal distribution. By this means, a safety factor and an associated reliability can be defined where the region of interest is within the bounds $\mu \pm \eta\sigma$.

The ability to define the statistics of the safety factor accurately depends on the type of density functions used for input variables and their mathematical relationship to the safety factor. By tradition, the density function of a variable is assumed to be a normal Gaussian distribution, in the absence of information to the contrary. When a variable is measured and a histogram is made, a large number of density functions can usually be fitted to it. The selection of one density function over another is usually complicated by lack of test data. The form of the density function is therefore chosen for simplicity of application as well as accuracy in fitting the data. The normal Gaussian distribution function or a variation of it, the log-normal function, is often selected for this reason.

Several advantages are evident in restricting interest to normal or log-normal distributions. They have simple mathematical properties and are defined completely by two parameters: the mean and standard deviation in the case of the normal distribution. Use of a normal distribution of the log of the variable allows fitting skewed distributions, as well as defining a variable that cannot be less than zero. When combining variables, it is useful to know that the distribution of a linear combination of normally distributed independent variables is normal. Similarly, the product of log-normally distributed variables is log-normally distributed.

Various approaches can be used in estimating the distribution function of the safety factor from the statistics of the constituent variables: the convolution method, the Monte-Carlo method, and the linear-perturbation method. The convolution method is the direct calculation of the desired probability density by an integration process, where the individual variables are mathematically defined probability densities (see the Appendix). Only when safety factor is in simple analytical form can the integration be performed explicitly. As the number and complexity of parameters increases, the mathematical difficulties far exceed the realm of practical design usage. It is therefore generally impractical for a designer to use the convolution technique.

The Monte-Carlo approach consists of randomly selecting values of parameters from their defined frequency distributions and repetitively calculating the safety factor to form a sample of the population. The number of calculations is independent of the number of parameters but depends, to some extent, on the desired confidence in the results. This method is useful for highly nonlinear problems where the parameters and safety factor may be significantly non-Gaussian. The limitation for a designer is the cost of the calculations.

The linear-perturbation method is presented as an engineering approach in estimating the safety factor distribution function. The method is restricted to nearly linear problems where the input parameters can be defined by normal distributions. Because of its simplicity, the method is a useful engineering tool in accessing a design by apportioning reliability in terms of safety factors. This is necessary before using a detailed structural optimization scheme for maximum performance at a specified reliability as discussed in Refs. 1 and 2.

Monte-Carlo Method

For nonlinear problems, the most practical method of analysis is the Monte-Carlo method. A mathematical model for safety factor is required in terms of the several input variables, where the statistics of the variables are known. By selecting sets of variables at random, values of safety factors are calculated. The set of calculations represents a random sample from which the reliability can be found.

The statistics of the variables must be known but need not be Gaussian or independent. If the probability densities of the variables are independent and Gaussian, it is necessary only to select numbers from a table of Gaussian random numbers and to calculate the values of the variables directly. If the variables are Gaussian but are correlated, one can use a matrix transformation technique for a definition of the variables.

If the variables are known to be non-Gaussian, it is best to use actual test data to define the statistics. The measurements can be tabulated and assigned an index. By selecting random numbers corresponding to the index, the variables are randomly selected in the proper manner.

If a random number has been obtained from any continuous probability density function, it is possible by a transformation to obtain a random number for the desired probability density function. Thus an alternative approach consists in finding a uniform random number from 0 to 1 and, by a transformation, obtaining a random number for the desired probability function. The transformation in this case is equating the uniform random number to the desired probability distribution function and solving for the variable defined by this distribution function.

This principle is used also in digital computers to obtain normally distributed random variables. Various techniques exist for random number generation for digital computers.[3] The following has been used often:

$$X_i = 2^{-\beta} r_i$$

where X_i is the pseudo-uniform random number; β is the word length of a binary machine; $i = 0, 1, 2, 3$; r_0 is an odd number (say, 1); $r_i + 1 = Kr_i \pmod{2^\beta}$; $K = 5^\alpha$; α = positive odd integer (say, 19); and $1 \leq r_i \leq 2^\beta$. The term $Kr_i \pmod{2^\beta}$ means that the most significant half of the product Kr_i is discarded when using a binary computer.

From this pseudo-random number from 0 to 1, it is possible to obtain a random number from any probability distribution function. This is done by setting the random number equal to the distribution function and solving for the corresponding value. For the normal distribution, the standardized normal random variable is obtained from tables, and the Gaussian random number of interest is calculated by simple algebra.

For example, the mean μ and standard deviation σ is known for a random variable P. Suppose that the uniform random number from 0 to 1 is X_u:

$$X_u = \int_{-\infty}^{X_N} \frac{1}{(2\pi)^{1/2}} e^{-X^2/2} \, dX$$

where $X_N = (P - \mu)/\sigma$. From tables, the value of X_N is found. Therefore, $P = X_N\sigma + \mu$. Having made a series of calculations, the estimate of reliability is made simply by dividing the number of safety factors greater than one by the total number of safety factors calculated. When the probability distributions are known for the input variables, the confidence that reliability R is greater than R_0 (where R_0 is the specified design reliability) is given by

$$C = 1 - \sum_{X=0}^{X_0} \frac{[N(1 - R_0)]Xe^{-(1-R_0)N}}{X!}$$

where N is the number of safety factors in the sample, and X_0 is the number of safety factors less than one. The basis for this concept of confidence is given in Ref. 6. The estimate X_0/N is a point estimate, whereas the estimate $R > R_0$ is an interval estimate. The designer should select the number of safety-factor calculations comprising the sample based on the desired reliability. If high reliability is desired, a large number of calculations are needed. In order to obtain an idea for the number of calculations comprising the sample based on the desired reliability and confidence, let

$$M > \frac{-\ln(1 - C)}{1 - R_0}$$

For example, if one desires 95% confidence with 0.999 reliability, the required number of calculations for safety factor is $M > 2996$.

One should always remember that, if little confidence exists in the input variables and the analytical function expressing the safety factor, not much confidence is obtained in the estimate of reliability. Intuitively, if each one of the input means has a confidence of 0.95, it is impossible to obtain an estimate of the mean of safety factor with a confidence greater than 0.95 no matter how large the sample of safety factors. Therefore, in most practical applications, N and M should be thought of as the implied number of calculations rather than the actual number of calculations comprising the sample.

The practical limitations of the Monte-Carlo method occur when large numbers of calculations are required of a sophisticated mathematical model that requires a great deal of computing time for each calculation. It is desirable, therefore, to restrict the problem to a simple mathematical model requiring as few calculations as possible, yet maintaining the desired accuracy. An example of application of the Monte-Carlo method is given in Ref. 4 for the design of a re-entry vehicle heat shield.

Linear-Perturbation Method

The linear-perturbation method assumes that the functional dependence of safety factor on input variables can be represented with sufficient accuracy by the linear terms of a Taylor series expansion:

$$SF = SF^* + \sum_{i=1}^{n} \left(\frac{\partial SF}{\partial p_i}\right)^* (p_i - p_i^*)$$

The reference values are those assumed in the region where failure is being defined. The safety factor is determined by an analytical function of the mode of failure.

The use of this equation implies that the partial derivatives are known. They may be obtained by taking the difference of numerically calculated results, varying one of the parameters at a time from a reference condition, and dividing the difference in safety factor by the corresponding difference in the parameter. As a general rule, variations in the parameter for this purpose should be small compared to the 1σ variations. The reference value p^* should always be in the region of interest, that is, in the region of variables that lead to failures or near-failures.

The mean value of safety factor can be found by

$$\overline{SF} = SF^* = + \sum_{i=1}^{n} \left(\frac{\partial SF}{\partial p_i}\right)^* (\bar{p}_i - p_i^*)$$

where the mean or expected value of safety factor is defined:

$$E\{SF\} \equiv \int_{SF} SF\, f(SF) d(SF)$$

If the parameters p_i are normally distributed, the safety factor must be normally distributed as well. Thus, the probability density function of SF is

$$f(SF) = \frac{1}{\sigma_{SF}\,(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{SF - \overline{SF}}{\sigma_{SF}}\right)^2\right]$$

with $\eta = (SF - \overline{SF})/\sigma_{SF}$. The probability distribution function and density function for the standardized normal random variable can be obtained in any engineering handbook.

If the input variables are statistically independent, the standard deviation of the safety factor is

$$\sigma_{SF}{}^2 = \sum_{i=1}^{n} \left(\frac{\partial SF}{\partial p_i}\right)^{*2} \sigma_{p_i}{}^2$$

When the variables (such as p_j and p_k) are dependent,

$$\sigma_{SF}{}^2 = \sum_{i=1}^{n} \left(\frac{\partial SF}{\partial p_i}\right)^{*2} \sigma_i{}^2 + 2 \sum_{<k}^{n} \left(\frac{\partial SF}{\partial p_j}\right)^{*} \left(\frac{\partial SF}{\partial p_k}\right)^{*} \sigma_{p_j p_k}$$

where

$$\sigma_{p_j p_k} = E\{p_j p_k\} - E\{p_j\}E\{p_k\}$$

This perturbation approach is based on the premise that the standard deviations of the input parameters are small when compared with the mean values. Thus, all of the input parameters as well as the corresponding safety factors will cluster within a small domain of the average case. The assumption implies that in the range of interest (usually $\pm 3\sigma$ variations in parameters) the influence of higher-order terms is negligible.

Heavy nonlinearities preclude the use of this method in a direct fashion. Nonlinearities can be handled in a combination approach with the Monte-Carlo method; the nonlinear aspects are handled separately by a Monte-Carlo technique, and the results are combined with those for the linear aspects into a perturbation analysis. An application of a combined solution has been the definition of the distribution function of re-entry heating by a Monte-Carlo method. The thermal environment is then used as a variable in a linear stress problem that is easily handled by the linear-perturbation method.

When a marked discontinuity occurs in one of the parameters but where the problem is nearly linear on either side of the discontinuity, one can use the perturbation method by making two analyses. By using reference conditions on either side of the discontinuity, the conditional probability is found that the given parameter falls above or below the critical value. Then, by putting the two results for the statistics of the safety factor together, the composite statistics for safety factor are found.

When handling nonlinear problems, the mean and standard deviations of safety factor as defined by the linear approach have no physical significance; instead, a normal distribution is obtained which gives a close approximation of the true probability of safety factor in the region corresponding to failure, provided that the reference conditions have been chosen by the forementioned criterion. The most practical estimate of reliability is obtained when the reference safety factor equals one (i.e., in the linear region that defines failure). If the problem is too nonlinear, the only alternative is to use the Monte-Carlo method.

When working with linear problems, the difference between the true mean and that calculated by the perturbation method is a measure of linearity. For example, if the safety factor is defined by the first-and second-order terms of a Taylor Series,

$$\overline{SF} = SF^* + \sum_{i=1}^{n} \left(\frac{\partial SF}{\partial p_i}\right)^* (\bar{p}_i - p_i^*) + \frac{1}{2}\sum_{i=1}^{n} \left(\frac{\partial^2 SF}{\partial p_i{}^2}\right)^* \times$$

$$[\sigma_{p_i}{}^2 + (\bar{p}_i - p_i^*)^2] + \frac{1}{2}\sum_{i \neq j} \frac{\partial^2 SF}{\partial p_i \partial p_j} \times$$

$$[(\bar{p}_i - p_i^*)(\bar{p}_j - p_j^*) + \sigma_{p_i}{}^2]$$

The additional terms represent the error one will obtain when estimating the true mean by the perturbation technique.

This method gives to the designer considerable insight into the importance of the design variables. The partial derivatives serve to indicate the effect when the designer changes a variable. Thus, it is possible to determine which variables must be analyzed exhaustively or tested to define their statistics, which variables must be given the most conservative design values, and which variables can be ignored, essentially. This method is a major contribution in preliminary design where detailed accuracy is unimportant.

Structural Optimization

For an engineering design estimation of safety factor, the convolution method is mathematically unwieldy, the Monte-Carlo method is the most accurate but extremely time-consuming and costly, and the linear perturbation method is a simple straightforward approach of acceptable engineering accuracy for near-linear problems. Therefore, it is proposed that the perturbation method be used for preliminary design optimization studies, whereas the Monte-Carlo method may be used for a check on the final design.

A. Assignment of Allowables

In the design of metal structures, "design practice" dictates the use of allowable stresses obtained from handbooks and the calculation of a safety factor greater than a specified value. An accepted source of strength data is MIL-HDBK-5, where data are furnished corresponding to 0.99 reliability and 0.90 reliability. Many other sources use 0.999 or 3σ reliability. If a given level of reliability is assigned to each variable (say strength, modulus, elongation, etc., as relevant in a given case of combined loading), intuitively the net safety factor resulting from a combination of variables has a reliability much greater than the assigned level of each variable. For design of metal structures, the resulting high reliability levels have been acceptable, since the associated weight has not been excessive. However, for plastics and composite structures the weight penalty is excessive, and a more realistic evaluation of design reliability must be made.

A common problem involving large variances in parameters is thermal stress produced in a composite plate. An example of this problem is shown in Table 1, where a plate of reinforced phenolic and magnesium is soaked at an elevated temperature. A comparison is made between standard design methods using a given reliability for each variable and the approach using a composite design reliability criterion.

The ability to optimize weight is clearly dependent on the method of defining a design criterion. The basis for comparison between materials for minimum weight must be made for equal performance, and equating performance can be done most realistically on a reliability basis. It must therefore be concluded that all structural optimization techniques must be subject to a composite design criterion, such as the reliability criterion discussed herein.

B. Selection of Materials

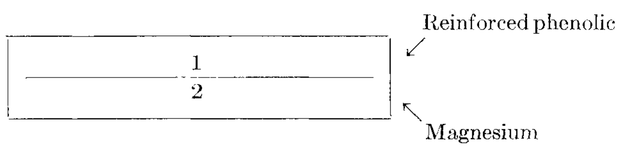
A common problem in structural optimization is the selection of a material to provide minimum weight. Having recognized that use of a reliability criterion is essential for comparison of materials having large variances in properties, the designer should define the probable modes of failure of the design with an associated desired reliability. For each mode of failure, the material is sized for the design reliability, and a weight comparison is made.

The design of composite structures requires compatability of materials under applied loads as well as thermal environments. A typical design problem is the bending of a composite plate where the substructure is fixed in geometry and material, but the protective coating (such as a heat shield) has large variances in properties. Table 2 presents such a problem, where two candidate materials are evaluated for the design of thermal protective material. The problem is to minimize the heat-shield weight for a given applied bending moment, assuming that the resultant thermal characteristics are satisfactory.

The phenolic material appears better than the epoxy on a basis of modulus of elasticity, but the strain allowable of the epoxy is better than the phenolic. Yet the results of the analysis show a considerable weight advantage for the epoxy material. The results reflect the variances existing in the materials (which have almost equal densities).

Table 1 Selection of design criterion

Mode of failure: thermal tensile stress in heat shield substructure.



$$SF = \frac{\epsilon_{\text{all}_1}}{\epsilon_1} \qquad \epsilon_1 = \frac{\alpha_1 \Delta T_1}{1 - \nu_1}\left[\frac{(\alpha_2\Delta T_2/\alpha_1\Delta T_1) - 1}{1 + [E_1t_1(1-\nu_2)/E_2t_2(1+\nu_1)]}\right]$$

Constants: $t_2 = 0.50$, $\nu_1 = 0.25$, $\nu_2 = 0.35$.

Variable	Mean	Std. dev.	0.90 rel.	0.99 rel.
$T_1 = T_2$	300	10	312.8	323.3
ϵ_{all_1}	0.0037	0.00086	0.0026	0.0017
α_1	4.0×10^{-6}	0.15×10^{-6}	3.81×10^{-6}	3.65×10^{-6}
α_2	15.27×10^{-6}	0.2×10^{-6}	15.52×10^{-6}	15.73×10^{-6}
E_1	1.4×10^6	0.2×10^6	1.144×10^6	0.927×10^6
E_2	4.75×10^6	0.2×10^6	5.006×10^6	5.216×10^6

Design criterion			Final design	
Variable reliability	SF reliability	Required SF	t_1	SF reliability
0.90	...	1.0	4.15	0.9918
0.99	...	1.0	6.67	0.9970
...	0.90	1.0	1.50	0.9000
...	0.99	1.0	3.85	0.9900
0.90	...	1.25	5.80	0.9911
...	0.90	1.25	2.45	0.9000

C. Sizing Structure for Minimum Weight

Having compared materials for a variety of modes of failures and then having selected the design material, the designer must determine the minimum-weight structure. This allows selection of combinations of various structural components for the various modes of failure, such that each mode achieves a desired reliability. Combining a small-perturbation technique with linear programming provides the designer a very simple and efficient means of evaluating the optimum combination of thicknesses for minimum weight.

Weight is assumed to be a linear function of the same form as safety factor:

$$W = W^* + \sum_{i=1}^{n} \left(\frac{\partial W}{\partial p}\right)_i (p_i - p_i^*)$$

Each mode of failure has a reliability defined by the number of standard deviations:

$$\eta_i = (\overline{SF}_i - 1)/\sigma_{SF_i}$$

The perturbation equations for all modes of failure are listed as constraints so that the reliability of each mode must be greater than a constant:

$$\sum_{\alpha=1}^{N} \left[\frac{1}{\sigma_{SF_i}} \left(\frac{\partial SF}{\partial p_\alpha}\right)_i^* (p_\alpha - p_\alpha^*)\right] + \left(\frac{SF^* - 1}{\sigma_{SF}}\right)_i \geq \eta_i$$

Besides these linear constraints for m modes of failure, there must be further constraints limiting the variables in the range of linearity for the safety factor function:

$$c_\alpha \leq p_\alpha \leq d_\alpha$$

where c_α and d_α are constants. This is the classic linear programming problem, which can be solved by the simplex method. Refer to Ref. 5 for standard simplex solutions.

The following example solution demonstrates the usefulness of this technique. For the purpose of illustration, the structural design of a re-entry vehicle may be defined by four independent modes of failure: 1) ascent: tension failure in metal; 2) re-entry maximum axial load: buckling failure of composite; 3) re-entry maximum lateral load: buckling failure of composite; and 4) maximum thermal stress: tension failure of heat shield. Each mode of failure is analyzed for a reference design that has the critical variables listed in Table 3. The linear-perturbation method is used to define the distribution function of safety factor for each mode of failure and the associated reliability as listed in Table 4. The structure that comprises the weight for optimization is described by four variables, as given in Table 5.

The minimization of weight will be effected by maximizing the function

$$f = -(490\, t_{asc} + 2000\, A_{stiff} + 490\, t_{H/S} + 1350\, t_{mag})$$

The constraints are written as

$$\eta_1 = 4.43 + \frac{20}{0.113}(t_{mag} - 0.08) + \frac{0.333}{0.113}(t_{asc} - 0.2)$$

$$\eta_2 = 2.32 + \frac{65}{0.177}(t_{mag} - 0.08) + \frac{22}{0.177}(A_{stiff} - 0.16)$$

$$\eta_3 = 0.138 + \frac{27.5}{0.29}(t_{mag} - 0.08) + \frac{12.4}{0.29}(A_{stiff} - 0.16)$$

$$\eta_4 = 3.71 - \frac{20}{0.59}(t_{mag} - 0.08) + \frac{35}{0.59}(t_{H/S} - 0.4)$$

Table 2 Selection of material for minimum weight

Mode of failure: tensile strain in heat shield due to bending moment.

$$SF = \frac{\epsilon_{all_1} I_{eff} E_1}{M\bar{X}}$$

Candidate materials:
a) Refrasil tape with silica-phenolic resin.
b) Refrasil fibers and silica powder with epoxy resin.

Material, all	Variable, M	Mean, 700 in.-lb	Std. dev., 30
2	E_2	4.75×10^6	0.2×10^6
2	t_2	0.5	0.001
a	ϵ_{all_1}	0.0037	0.00086
a	E_1	2.09×10^6	0.28×10^6
b	ϵ_{all_1}	0.0056	0.0006
b	E_1	0.99×10^6	0.19×10^6

Reliability	Required SF	Heat-shield material	Heat-shield thickness	Heat-shield weight
0.98	1.0	a	0.478	80.8
0.98	1.0	b	0.210	27.3

Table 3 Re-entry vehicle variables

Condition	Variable	Mean	Std. dev.	$\partial SF/\partial p$
Ascent	F_T	2.25×10^4	0.12×10^4	0.0000666
	P_{asc}	1.20×10^4	0.06×10^4	−0.000117
	t_{mag}	0.080	0.002	20.0
	t_{asc}	0.2	0.002	0.333
Re-entry max. long.	P_{long}	25	1.5	−0.0533
	E_{mag}	5.85×10^6	0.1×10^6	0.2×10^{-6}
	t_{mag}	0.080	0.002	65
	A_{stiff}	0.16	0.004	22
Re-entry max. lat.	P_{lat}	16	1.5	−0.057
	E_{mag}	5.85×10^6	0.1×10^6	0.2×10^{-6}
	t_{mag}	0.080	0.002	27.5
	A_{stiff}	0.16	0.004	12.4
Thermal stress	$\alpha_{H/S}$	8.9×10^{-6}	0.7×10^{-6}	0.585×10^6
	α_{mag}	15.27×10^{-6}	0.2×10^{-6}	-0.45×10^6
	ΔT	350	30	−0.00834
	$E_{H/S}$	0.99×10^6	0.19×10^6	1.525×10^{-6}
	E_{mag}	4.75×10^6	0.2×10^6	-0.05×10^{-6}
	$\epsilon_{H/S}$	0.005	0.0002	650
	$t_{H/S}$	0.4	0.002	35
	t_{mag}	0.080	0.002	−20

Table 4 Re-entry vehicle reference design

Failure condition	$\overline{SF}$	σ_{SF}	η
Ascent: tension failure in metal	1.5	0.113	4.43
Re-entry max. axial: buckling	1.41	0.177	3.32
Re-entry max. lateral: buckling	1.04	0.290	0.138
Max. thermal stress: tension in heat shield	3.19	0.59	3.71

Further constraints to be imposed are that no mode of failure shall have a reliability less than that associated with $\eta = 3$ and that manufacturing limits on sizes are $t_{asc} \geq 0.10$, $A_{stiff} \geq 0.10$, $t_{mag} \geq 0.020$, $t_{H/S} \geq 0.020$.

By the simplex method, the structural variables are found to be $t_{asc} = 0.10$, $A_{stiff} = 0.10$, $t_{mag} = 0.137$, $t_{H/S} = 0.42$, and the total structural weight is 267.7 lb. Thus the proper combination of components in a composite structure is found, producing minimum weight but maintaining a given reliability for each mode of failure.

By an obvious extension of this approach, the case of a given combined reliability can be handled. The result is a minimum weight for the combined reliability, as well as an optimum reliability apportionment to the several structural elements.

Concluding Remarks

The application of a reliability criterion, when used with a correct analysis of failure modes, results in a design with a known degree of conservatism. The criterion is preferable over standard design practice for complicated combinations of environments acting sequentially and simultaneously. The methods presented can also be used for calculating reliability of a given design.

Although one can devise elaborate schemes to define uncertainties and randomness, the designer, when estimating a safety factor, must recognize the limitations involved. Very rarely can one accurately define the probability density functions of the design parameters. Because of the necessary lack of these, the estimate of reliability should be simple in order to obtain answers quickly and efficiently. The techniques presented here in the use of the linear-perturbation method are well suited to the limitations of preliminary design. It is questionable whether more sophisticated statistical techniques are meaningful to a preliminary designer or if such techniques can be readily adapted within practical limitations. The development and use of refined analyses are more desirable during detail design where the Monte-Carlo technique is the most practical approach.

Appendix: Distribution of the Quotient of Two Normally Distributed Random Variables

The convolution method is used to define the mean, variance, and probability density function of $Y = X_2/X_1$, where X_2 and X_1 are independent.

The probability density function is the derivative of the probability distribution function:

$$\frac{\partial^2 F(X_1, X_2)}{\partial X_1 \partial X_2} = f_{X_1}(X_1)\cdot f_{X_2}(X_2)$$

$$F_{X_1, X_2}(A, B) = P\{X_1 \leq A, X_2 \leq B\}$$

The probability that $Y < y$ is the integral of the probability density function of X_1 and X_2 over all values where $X_2/X_1 < y$. Thus,

$$F_Y(y) = \iint_{\{X_1, X_2: X_2/X_1 < y\}} f_{X_1}(X_1)\cdot f_{X_2}(X_2) dX_2 dX_1$$

$$f_Y(y) = \int_{-\infty}^{+\infty} |X_1| f_{X_1}(X_1) f_{X_2}(yX_1) dX_1$$

When X_1 and X_2 are independent standardized normal random variables,

$$f_Y(y) = 2\int_0^{+\infty} X_1 \frac{1}{(2\pi)^{1/2}} e^{-X_1^2/2} \frac{1}{(2\pi)^{1/2}} e^{-(X_1 y)^2/2} dX_1$$

which yields

$$f_Y(y) = 1/\pi(1 + y^2)$$

$$F_Y(y) = (1/\pi) \tan^{-1} y + \tfrac{1}{2}$$

By convolution the function defined is the familiar Cauchy distribution, which has no standard deviation.

In the more general case where X_1 and X_2 are independent normal random variables with known means and standard deviations, the mean as well as the standard deviation of Y does not exist. If we assume that X_1 has a truncated normal distribution where the density function is truncated at $\mu_1 - \eta\sigma_1$ and $\mu_1 + \eta\sigma_1$, an estimate of the mean μ_Y, and variance σ_Y^2 can be made assuming that η is sufficiently large:

$$\mu_Y = \frac{\mu_2}{\mu_1}\left[1 + \frac{\sigma_1^2}{\mu_1^2} + \frac{3\sigma_1^4}{\mu_1^4}\right]$$

Thus the mean of Y can be thought of as μ_2/μ_1 plus an error for the truncated distribution where X_1 must always be greater than zero. The variance is defined by

$$\sigma_Y^2 = \frac{\sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2}{\mu_1^2} + \varphi$$

$$\varphi \leq \frac{3\sigma_1^2\sigma_2^2}{\mu_1^4} + \frac{8\mu_2^2\sigma_1^4}{\mu_1^6}$$

Table 5 Re-entry vehicle reference weight

Component	Weight	Structural variable, p	$\partial w/\partial p$
Ascent shield	27	Ascent cover, t_{asc}	490
Re-entry shield	54	Shell stiffener, A_{stiff}	2000
Shell structure	196	Re-entry heat shield, $t_{H/S}$	490
Internal structure	73	Shell, t_{mag}	1350
	350 lb		

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